Exercise Sheet #6

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P1. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space and $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ a random variable, i.e., $\forall A \in \mathcal{B}(\mathbb{R}), X^{-1}(A) \in \mathcal{F}$. Prove that the distribution function of X, $F_X : \mathbb{R} \to [0,1]$ defined by $F(x) = \mathbb{P}(X \leq x)$, determines the measure induced by X, $\mathbb{P}_X : \mathcal{B}(\mathbb{R}) \to [0,1]$ defined by $\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A))$.

P2. Consider $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$ where λ is the Lebesgue measure. Let μ be a measure on $\mathcal{B}(\mathbb{R})$ that satisfies the following conditions:

i) For all $A \in \mathcal{B}(\mathbb{R})$ and $x \in \mathbb{R}$: $\mu(A) = \mu(A + x)$.

ii) $0 < \mu((0,1]) < \infty$.

Show that there exists $\alpha > 0$ such that $\mu = \alpha\lambda$.

P3. In this exercise, we prove that there exists a Lebesgue non-measurable subset of $\mathbb{R}$. For this, we define an equivalence relation on $[0,1)$ by $x\mathcal{R}y$ if $y - x \in \mathbb{Q}$. By axion of choice, let $E \subseteq [0,1)$ be a set containing exactly one representative of each equivalence class, and for each $t \in \mathbb{Q} \cap [0,1]$ let $E_t = \{x + t \bmod 1 \mid x \in E\} \subseteq [0,1)$.

(a) Show that the sets $(E_t)_{t \in \mathbb{Q} \cap [0,1]}$ are pairwise disjoint.

(b) Show that

$$\bigsqcup_{t \in \mathbb{Q} \cap [0,1]} E_t = [0,1).$$

(c) Assume by contradiction that E is Lebesgue measurable. Show that for every $t \in \mathbb{Q} \cap [0,1)$, E_t is Lebesgue measurable and $\lambda(E_t) = \lambda(E)$.

(d) Conclude by arriving to a contradiction.

P4. Consider λ as the Lebesgue measure on $\mathbb{R}$, and let A be a Lebesgue measurable set. Prove that if $\lambda(A) > 0$, then A contains a non-measurable set.